

Yang–Baxter maps with first–degree–polynomial 2×2 Lax matrices

Theodoros E. Kouloukas and Vassilios G. Papageorgiou *

Department of Mathematics, University of Patras,
GR-265 00 Patras, Greece

Abstract

A family of non-parametric Yang–Baxter (YB) maps is constructed by re-factorization of the product of two 2×2 matrix polynomials of first degree. These maps are Poisson with respect to the Sklyanin bracket. For each Casimir function a parametric Poisson YB map is generated by reduction on the corresponding level set. By considering a complete set of Casimir functions symplectic multi-parametric YB maps are derived. These maps are quadrirational with explicit formulae in terms of matrix operations. Their Lax matrices are, by construction, 2×2 first-degree-polynomial in the spectral parameter and are classified by Jordan normal form of the leading term. Non-quadrirational parametric YB maps constructed as limits of the quadrirational ones are connected to known integrable systems on quad-graphs.

1 Introduction

The question of finding set-theoretical solutions of the quantum Yang–Baxter equation was first suggested by Drinfeld in [5]. Certain examples of such solutions had already appeared in the relevant literature by Sklyanin [15]. The dynamical aspects of these solutions were studied by Veselov in [17] where the short term “Yang–Baxter maps” was proposed for them. Recent results [10, 11] connect these solutions with integrable equations on quad-graphs through symmetry–reduction. Actually the connection between the YB relation for maps and the *multidimensional consistency* property [9, 4] for discrete equation on quad-graphs was already noticed by Adler, Bobenko and Suris (see concluding remarks of [1]). They also gave a classification of YB maps on $\mathbb{CP}^1 \times \mathbb{CP}^1$ in [2]. Weinstein and Xu [19] found a way of constructing YB maps (classical solutions of the quantum YB equation, in their terminology) using the theory of Poisson–Lie groups. Their method was generalised in [8]. The algebraic theory of YB maps was developed by Etingof, Schedler and Soloviev [6]. It seems though that dressing transformations connected to soliton equations and associated constructions involving loop groups are giving easily many low dimensional examples of YB maps as well as the most simple and fundamental parametric one i.e. Adler’s map. The construction of the latter in [13] was given by Hamiltonian reduction of the loop group $LGL(2, \mathbb{R})$ equipped with the Sklyanin bracket [14]. For a review on YB maps one can look at [18].

Based on these ideas we present in this work a construction of symplectic, parametric YB maps on $\mathbb{C}^2 \times \mathbb{C}^2$ with 2×2 first-degree-polynomial Lax matrices. In section 2 we give the necessary definitions and notation. Section 3 contains the construction of a non-parametric quadrirational YB map from a re-factorization procedure. The proof of its YB property is put in an appendix. This map is Poisson with respect to the Sklyanin bracket presented in section 4. A reduction procedure to symplectic leaves is also applied in order to obtain symplectic parametric YB maps and their corresponding Lax matrices. A classification is provided by Jordan normal forms. In section 5 non-quadrirational YB maps are

*E-Mail: tkoulou@master.math.upatras.gr, vassilis@math.upatras.gr

derived as limits of the quadrirational ones of the previous section. We finally conclude in section 6 giving some perspectives for future work.

2 Yang-Baxter Maps and Lax Matrices

A *set-theoretic solution of the Quantum Yang-Baxter equation* [5], or just a *Yang-Baxter (YB) Map* [17] is a map $R : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$, where \mathcal{X} is any set, which satisfies the equation:

$$R_{23}R_{13}R_{12} = R_{12}R_{13}R_{23} \quad (1)$$

where R_{ij} for $i, j = 1, \dots, 3$ is the map that acts as R on the i and j factors of $\mathcal{X} \times \mathcal{X} \times \mathcal{X}$ and identically on the rest. In various examples of YB maps, e.g. maps arising from geometric crystals [7], the set \mathcal{X} has the structure of an algebraic variety and R is a birational isomorphism. We are also concerned with birational YB maps here as well. A Yang-Baxter map $R : (x, y) \mapsto (u, v) = (u(x, y), v(x, y))$ is called *quadrirational* [2] if the maps $u(\cdot, y) : \mathcal{X} \rightarrow \mathcal{X}$ and $v(x, \cdot) : \mathcal{X} \rightarrow \mathcal{X}$ are bijective rational maps.

A *parametric YB map* is a YB map

$$R : ((x, \alpha), (y, \beta)) \mapsto ((u, \alpha), (v, \beta)) = (u(x, \alpha, y, \beta), v(x, \alpha, y, \beta))$$

where $x, y \in \mathcal{X}$ and the parameters $\alpha, \beta \in \mathbb{C}^n$. It is useful to keep the parameters separately and denote $R(x, \alpha, y, \beta)$ by $R_{\alpha, \beta}(x, y)$. A *Lax Matrix* for this map is a matrix $L(x, \alpha, \zeta)$ that depends on the point x , the parameter α and a spectral parameter ζ (we usually denote it just by $L(x; \alpha)$), such that

$$L(u; \alpha)L(v; \beta) = L(y; \beta)L(x; \alpha), \quad (2)$$

for any $\zeta \in \mathbb{C}$. Here we have adopted the definition of a Lax matrix from [16] but we have to notice that this definition does not imply necessary that equation (2) is equivalent to $(u, v) = R_{\alpha, \beta}(x, y)$.

We can represent any parametric YB map with an elementary quadrilateral like in Fig.1.

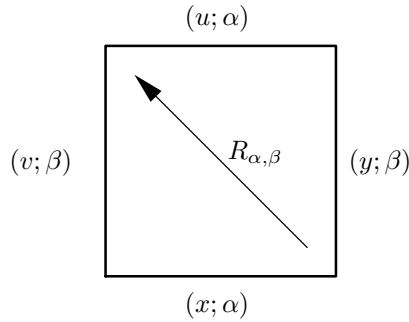


Fig.1 A map assigned to the edges of a quadrilateral

Let $R_{23}R_{13}R_{12}(x, y, z) = (x'', y'', z'')$ and $R_{12}R_{13}R_{23}(x, y, z) = (\tilde{x}, \tilde{y}, \tilde{z})$. We can represent these maps as chains of maps at the faces of a cube like in Fig.2.

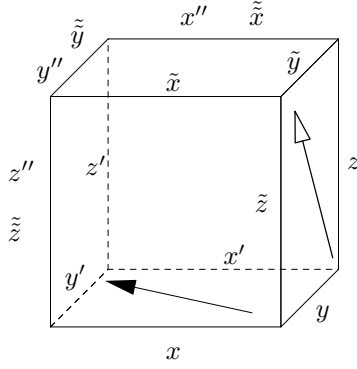


Fig.2 Representation of the Yang-Baxter property

The first map corresponds to the composition of the down, back, left faces, while the second one to the right, front and upper faces. All the parallel edges to the x (resp. y, z) axis carry the parameter α (resp. β, γ). So Eq.(1) assures that $x'' = \tilde{x}$, $y'' = \tilde{y}$, $z'' = \tilde{z}$.

The proof of the following proposition based on the associativity property of matrix multiplication appears essentially in [17].

Proposition 2.1. *Let $u = (x, y)$, $v = v(x, y)$ and $A(x; \alpha)$ a matrix depending on a point x , a parameter α and a spectral parameter ζ , such that $A(u; \alpha)A(v; \beta) = A(y; \beta)A(x; \alpha)$.*

If the equation

$$A(\hat{x}; \alpha)A(\hat{y}; \beta)A(\hat{z}; \gamma) = A(x; \alpha)A(y; \beta)A(z; \gamma) \quad (3)$$

implies that $\hat{x} = x$, $\hat{y} = y$ and $\hat{z} = z$, then the map $R_{\alpha, \beta}(x, y) = (u, v)$ is a parametric Yang-Baxter map with Lax matrix $A(x; \alpha)$.

Proof. Let A be a matrix with the above properties. The cubic representation of Fig.2 at the down, back and left faces give $A(y; \beta)A(x; \alpha) = A(x'; \alpha)A(y'; \beta)$, so $A(z; \gamma)A(y; \beta)A(x; \alpha) = (A(z; \gamma)A(x'; \alpha))A(y'; \beta) = A(x''; \alpha)(A(z'; \gamma)A(y'; \beta)) = A(x''; \alpha)A(y''; \beta)A(z''; \gamma)$. Similarly from the right, front and upper faces we get $A(z; \gamma)A(y; \beta)A(x; \alpha) = A(\tilde{x}; \alpha)A(\tilde{y}; \beta)A(\tilde{z}; \gamma)$. So we have that

$$A(x''; \alpha)A(y''; \beta)A(z''; \gamma) = A(\tilde{x}; \alpha)A(\tilde{y}; \beta)A(\tilde{z}; \gamma)$$

which implies that $x'' = \tilde{x}$, $y'' = \tilde{y}$, $z'' = \tilde{z}$. i.e. the Yang-Baxter property (1). \square

3 Yang-Baxter maps from matrix re-factorization

Our aim is to find the YB maps corresponding to those 2×2 Lax matrices that are first-degree matrix polynomials with respect to the spectral parameter. So we consider the set \mathcal{L} of 2×2 polynomial matrices of the form $L(\zeta) = A - \zeta B$, $\zeta \in \mathbb{C}$. Let us fix a constant matrix B in $GL_2(\mathbb{C})$. We denote by i_B the immersion $i_B : GL_2(\mathbb{C}) \rightarrow \mathcal{L}$ with $i_B(A) = A - \zeta B$ and by p_A the polynomial

$$p_A(\zeta) := \det(A - \zeta B) \equiv f_2(A)\zeta^2 - f_1(A)\zeta + f_0(A)$$

where the scalar functions f_i , $i = 0, 1, 2$ are given by

$$f_2(A) = \det B, \quad f_1(A) = \det B \text{Tr}(AB^{-1}), \quad f_0(A) = \det A.$$

We also define the functions $P_i : GL_2(\mathbb{C}) \times GL_2(\mathbb{C}) \rightarrow GL_2(\mathbb{C})$ for $i = 1, 2$, with

$$P_1(X, Y) = f_2(X)(YB + BX) - f_1(X)B^2, \quad (4)$$

$$P_2(X, Y) = f_2(X)YX - f_0(X)B^2. \quad (5)$$

Let X, Y be generic elements of $GL_2(\mathbb{C})$. We want to find $U = U(X, Y)$, $V = V(X, Y)$, such that the equation

$$i_B(U)i_B(V) = i_B(Y)i_B(X) \quad (6)$$

holds for any $\zeta \in \mathbb{C}$. First we notice that this equation admits the trivial solution $U = Y$, $V = X$. The next proposition give us a second more interesting solution.

Proposition 3.1. *Let $X, Y \in GL_2(\mathbb{C})$ such that $\det P_1(X, Y) \neq 0$. Then there are unique $U = U(X, Y)$ and $V = V(X, Y)$ in $GL_2(\mathbb{C})$, where*

$$U(X, Y) = P_2(X, Y)P_1(X, Y)^{-1}B, \quad V(X, Y) = B^{-1}(YB + BX - UB), \quad (7)$$

that satisfy equation (6) and the constraint $\det(U - Y) \neq 0$ (equivalently $\det(V - X) \neq 0$). The map $R(X, Y) = (U(X, Y), V(X, Y))$ is a (non-parametric) quadrirational Yang-Baxter map such that $f_i(U) = f_i(X)$ and $f_i(V) = f_i(Y)$ for $i = 0, 1, 2$.

Proof. Equation (6) is equivalent with the system:

$$UV = YX, \quad UB + BV = YB + BX. \quad (8)$$

If we write the first equation as $UB^{-1}BV = YX$, replace BV from the second one and after some simple algebra, we have that

$$UB^{-1}(Y - U) = (Y - U)XB^{-1}, \quad (X - V)B^{-1}V = B^{-1}Y(X - V). \quad (9)$$

These two relations show that if there exist a solution of (6) with $\det(U - Y) \neq 0$ (equivalently $\det(V - X) \neq 0$), then the matrices UB^{-1} , $B^{-1}V$ must be similar to the matrices XB^{-1} and $B^{-1}Y$ respectively. So $p_U(\zeta) = p_X(\zeta)$ and $p_V(\zeta) = p_Y(\zeta)$.

Suppose that U, V is a solution of equation (6) with $\det(U - Y) \neq 0$. Cayley-Hamilton theorem states that $p_U(UB^{-1}) = 0$. Since $p_U(\zeta) = p_X(\zeta)$, we get that $p_X(UB^{-1}) = 0$ i.e.

$$f_2(X)(UB^{-1})^2 - f_1(X)(UB^{-1}) = -f_0(X)I. \quad (10)$$

Also system (8) gives : $(UB^{-1})^2B^2 = UB^{-1}(YB + BX) - YX$. So by solving (10) with respect to U and substituting (8) we obtain (7). Here we have assumed that $\det P_1(X, Y) \neq 0$ for the generic elements X, Y (see remark 3.2).

So far we have proved that if a solution exists with $\det(U - Y) \neq 0$, then it will be unique and will have the form (7). We still have to check that these U, V satisfy system (8). The second equation of the system is obviously satisfied. Now (7) implies that $UB^{-1}(f_2(X)(YB + BX) - f_1(X)B^2) = (f_2(X)YX - f_0(X)B^2)$ and $YB + BX = UB + BV$, so

$$(f_2(X)(UB^{-1})^2 - f_1(X)UB^{-1} + f_0(X)I)B^2 = f_2(X)(YX - UV). \quad (11)$$

Moreover we can write

$$\begin{aligned} P_2(X, Y) &= P_2(X, Y) + P_1(X, Y)B^{-1}X - P_1(X, Y)B^{-1}X = \\ &= P_1(X, Y)B^{-1}X - B^2(f_2(X)(B^{-1}X)^2 - f_1(X)B^{-1}X + f_0(X)I) \end{aligned}$$

so $P_2(X, Y) = P_1(X, Y)B^{-1}X$. (Here we used again Cayley-Hamilton theorem). Thus we have the following equivalent expression for U :

$$UB^{-1} = P_1(X, Y)B^{-1}XB^{-1}(P_1(X, Y)B^{-1})^{-1} \quad (12)$$

Which means that $f_i(X) = f_i(U)$ for $i = 0, 1, 2$. So from (11) and the Cayley-Hamilton theorem we finally derive that $UV = YX$.

We define now the map

$$R(X, Y) = (U, V) \quad (13)$$

where U, V is the solution of (8) given by (7). The quadrirationality of this map is already proven since (9) yields V, X , in terms of Y, U :

$$VB^{-1} = (U - Y)^{-1}YB^{-1}(U - Y), \quad XB^{-1} = (Y - U)^{-1}UB^{-1}(Y - U).$$

from which we had already that $f_i(X) = f_i(U)$ and $f_i(Y) = f_i(V)$. \square

We will refer to (13) as the *general Yang-Baxter map associated with the matrix B* and denote it by \mathcal{R}_B .

Remark 3.2. The functions $U(X, Y) = U, V(X, Y) = V$ of (7) are rational and of course not defined everywhere on $\mathbb{C}^4 \times \mathbb{C}^4$ but just in an open and dense domain $I \subset \mathbb{C}^4 \times \mathbb{C}^4$ defined by the restriction $\det P_1(X, Y) \neq 0$. Proposition 3.1 holds in this domain.

4 Poisson Structure and Reduction

We equip the manifold \mathcal{L} with the Sklyanin bracket [14]. We will show how we can reduce the general Yang-Baxter Map to Poisson submanifolds in order to obtain Poisson parametric Yang-Baxter Maps on these submanifolds. This reduction is possible due to the fact that the Casimir functions of this structure are exactly the invariant functions f_i that defined in the previous section. If the corresponding level set is a symplectic submanifold then the reduced YB map is symplectic.

4.1 Poisson Structure on \mathcal{L}

The Sklyanin bracket between the variables of a matrix polynomial $L(\zeta)$ of any degree is given by the formula :

$$\{L(\zeta) \otimes L(\eta)\} = \left[\frac{r}{\zeta - \eta}, L(\zeta) \otimes L(\eta)\right] \quad (14)$$

Here r denotes the permutation matrix: $r(x \otimes y) = y \otimes x$. The restriction of this bracket on the submanifold \mathcal{L} , of functions of the form $L(\zeta) = A - \zeta B$, with

$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ and $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$, is given by the Poisson structure anti-symmetric matrix :

$$J_B(A) = \begin{pmatrix} 0 & -a_2b_1 + a_1b_2 & a_3b_1 - a_1b_3 & a_3b_2 - a_2b_3 \\ * & 0 & a_4b_1 - a_1b_4 & a_4b_2 - a_2b_4 \\ * & * & 0 & -a_4b_3 + a_3b_4 \\ * & * & * & 0 \end{pmatrix} \quad (15)$$

where $J_B(A)_{ij}$ denotes the bracket $\{a_i - \zeta b_i, a_j - \zeta b_j\}$, for $i, j = 1, \dots, 4$.

First we notice that matrix B belongs to the center of this Poisson algebra (so $J_B(A)_{ij}$ is just $\{a_i, a_j\}$). As in [13] we restrict to the level set for $B = \text{Constant}$ and denote this Poisson submanifold by \mathcal{L}_B . The Casimir functions for the Poisson structure (15) on \mathcal{L}_B are:

$$f_0(A) = \det A, \quad f_1(A) = a_1b_4 + a_4b_1 - a_3b_2 - a_2b_3$$

These are the coefficients of $\det(A - \zeta B)$ and agree with f_0, f_1 defined in section 3. By this notation the general YB map that we constructed in the previous section is a non-parametric YB map $\mathcal{R}_B : \mathcal{L}_B \times \mathcal{L}_B \rightarrow \mathcal{L}_B \times \mathcal{L}_B$.

We can extend the Poisson bracket of \mathcal{L}_B to the Cartesian product $\mathcal{L}_B \times \mathcal{L}_B$ as follows :

$$\{x_i, x_j\} = J_B(X)_{ij}, \quad \{y_i, y_j\} = J_B(Y)_{ij}, \quad \{x_i, y_j\} = 0, \quad (16)$$

for any $(X - \zeta B, Y - \zeta B) \in \mathcal{L}_B \times \mathcal{L}_B$ where x_i, x_j, y_i, y_j for $i = 1, \dots, 4$ are the elements of the matrices X, Y respectively and J_B the matrix of the Poisson structure (15).

Proposition 4.1. *The general YB map $\mathcal{R}_B : \mathcal{L}_B \times \mathcal{L}_B \rightarrow \mathcal{L}_B \times \mathcal{L}_B$ is a Poisson map.*

Proof. A detailed computation shows that the Poisson bracket between the entries of U, V defined by (7) is:

$$\{u_i, u_j\} = J_B(U)_{ij}, \quad \{v_i, v_j\} = J_B(V)_{ij}, \quad \{u_i, v_j\} = 0,$$

for $i = 1, \dots, 4$. □

Remark 4.2. Let \mathfrak{g} be the four dimensional four parametric Lie algebra, with basis $\{e_1, e_2, e_3, e_4\}$, defined by

$$\begin{aligned} [e_1, e_2] &= b_2 e_1 - b_1 e_2, & [e_1, e_3] &= -b_3 e_1 + b_1 e_3, & [e_1, e_4] &= -b_3 e_2 + b_2 e_3, \\ [e_2, e_3] &= b_1 e_4 - b_4 e_1, & [e_2, e_4] &= -b_4 e_2 + b_2 e_4, & [e_3, e_4] &= b_4 e_3 - b_3 e_4, \end{aligned}$$

where b_i for $i = 1, \dots, 4$ are free parameters. Then the Poisson structure (15) on \mathcal{L}_B coincides with the corresponding Lie-Poisson structure on the dual \mathfrak{g}^* :

$$\{F, G\}_{L-P}(x) = \langle x, [d_x F, d_x G] \rangle, \quad x \in \mathfrak{g}^*, \quad F, G \in C^\infty(\mathfrak{g}^*).$$

4.2 Parametric YB maps and Lax Matrices

Let $A - \zeta B$ be a generic element of \mathcal{L}_B and a_{ij} an element of A with $\frac{\partial f_1}{\partial a_{ij}} \neq 0$. If we set $f_0(A) = c$, then there exist a function F_0 such that $F_0(a_1, a_2, a_3, c) = a_{ij}$, where a_1, a_2, a_3 here and below denote the remaining three entries of A . We denote by $L'_0(a_1, a_2, a_3; c)$ the matrix that is derived by replacing the a_{ij} element of A by $F_0(a_1, a_2, a_3, c)$, and by $L_0(a_1, a_2, a_3; c)$ the matrix $i_B(L'_0(a_1, a_2, a_3; c))$. We also define the projection $p_{ij} : GL_2(\mathbb{C}) \rightarrow \mathbb{C}^3$ to the elements of a matrix except of the ij element and the function $P : GL_2(\mathbb{C}) \times GL_2(\mathbb{C}) \rightarrow \mathbb{C}^3 \times \mathbb{C}^3$ with $P(X, Y) = (p_{ij}(X), p_{ij}(Y))$.

In a similar way if a_{ij} is an element of A such that $\frac{\partial f_1}{\partial a_{ij}} \neq 0$, we define the matrix $L'_1(a_1, a_2, a_3; c)$ by setting $f_1(A) = c$, the matrix $L_1(a_1, a_2, a_3; c) = L'_1(a_1, a_2, a_3; c) - \zeta B$ and the corresponding projection.

Let also a_{ij}, a_{kl} be two elements of A such that $\det [\frac{\partial(f_0, f_1)}{\partial(a_{ij}, a_{kl})}] \neq 0$. If we set $f_0(A) = c_1$ and $f_1(A) = c_2$, there exist two functions G_0, G_1 such that $G_0(a_1, a_2, c_1, c_2) = a_{ij}$ and $G_1(a_1, a_2, c_1, c_2) = a_{kl}$. We denote by $L'(a_1, a_2; c_1, c_2)$ the matrix that is obtained by replacing the a_{ij} and the a_{kl} elements of A by $G_0(a_1, a_2, c_1, c_2)$ and $G_1(a_1, a_2, c_1, c_2)$ respectively and $L(a_1, a_2; c_1, c_2) = i_B(L'(a_1, a_2; c_1, c_2))$. We also define the projection $q_{ij,kl} : GL(2) \rightarrow \mathbb{C}^2$ to the elements of a matrix except the ij and kl elements and the function $Q : GL(2) \times GL(2) \rightarrow \mathbb{C}^2 \times \mathbb{C}^2$ with $Q(X, Y) = (q_{ij,kl}(X), q_{ij,kl}(Y))$.

Since f_0, f_1 are Casimir functions the sets

$$\begin{aligned} \mathcal{P}_{B_0}(c) &= \{L_0(a_1, a_2, a_3; c) \in \mathcal{L}_B \mid f_0(L_0(a_1, a_2, a_3; c)) = c\}, \\ \mathcal{P}_{B_1}(c) &= \{L_1(a_1, a_2, a_3; c) \in \mathcal{L}_B \mid f_1(L_1(a_1, a_2, a_3; c)) = c\} \end{aligned}$$

are Poisson submanifolds of \mathcal{L}_B and the sets

$$\mathcal{S}_B(c_1, c_2) = \{L(a_1, a_2; c_1, c_2) \in \mathcal{L}_B \mid f_0(L(a_1, a_2; c_1, c_2)) = c_1, f_1(L(a_1, a_2; c_1, c_2)) = c_2\}$$

are two dimensional symplectic submanifolds of \mathcal{L}_B with the reductive symplectic form from (15).

Now we return to the general YB map $\mathcal{R}_B(X, Y) = (U(X, Y), V(X, Y))$, where $U(X, Y)$ and $V(X, Y)$ are defined by (7). We can reduce this map to parametric YB maps on the sets $\mathcal{P}_{B_0} \times \mathcal{P}_{B_0}, \mathcal{P}_{B_1} \times \mathcal{P}_{B_1}$ and $\mathcal{S}_B \times \mathcal{S}_B$.

Proposition 4.3. *The maps $R_{\alpha, \beta}^0, R_{\alpha, \beta}^1$ defined by*

$$\begin{aligned} R_{\alpha, \beta}^0(x, y) &= P \circ \mathcal{R}_B(L'_0(x_1, x_2, x_3; \alpha), L'_0(y_1, y_2, y_3; \beta)) \\ R_{\alpha, \beta}^1(x, y) &= P \circ \mathcal{R}_B(L'_1(x_1, x_2, x_3; \alpha), L'_1(y_1, y_2, y_3; \beta)), \end{aligned}$$

where $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$, are quadrirational Poisson parametric Yang-Baxter maps on $\mathbb{C}^3 \times \mathbb{C}^3$ with parameters α , β and Lax matrices $L_0(x_1, x_2, x_3; \alpha)$, $L_1(x_1, x_2, x_3; \alpha)$ respectively. The map

$$R_{\alpha, \beta}((x_1, x_2), (y_1, y_2)) = Q \circ \mathcal{R}_B(L'(x_1, x_2; \alpha_1, \alpha_2), L'(y_1, y_2; \beta_1, \beta_2)), \quad (17)$$

is a quadrirational symplectic parametric Yang-Baxter map on $\mathbb{C}^2 \times \mathbb{C}^2$ with vector parameters $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2)$ and Lax matrix $L(x_1, x_2; \alpha_1, \alpha_2)$.

Proof. The construction of the matrices $L'_0(x_1, x_2, x_3; \alpha)$ and $L'_0(y_1, y_2, y_3; \beta)$ implies that there are $X, Y \in GL_2(\mathbb{C})$ such that $L'_0(x_1, x_2, x_3; \alpha) = X$, $L'_0(y_1, y_2, y_3; \beta) = Y$ and $f_0(X) = \alpha$, $f_0(Y) = \beta$. Then

$$\mathcal{R}_B(L'_0(x_1, x_2, x_3; \alpha), L'_0(y_1, y_2, y_3; \beta)) = \mathcal{R}_B(X, Y) = (U, V),$$

where $U = U(X, Y)$, $V = V(X, Y)$ are defined by (7). This is a Poisson YB map such that $f_0(U) = \alpha$, $f_0(V) = \beta$. So $U = L'_0(u_1, u_2, u_3; \alpha)$, $V = L'_0(v_1, v_2, v_3; \beta)$. The projection $P(U, V)$ give us the corresponding elements $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$. The Yang Baxter property of this map, as well as the quadrirationality, are immediately derived from the YB property and the quadrirationality of \mathcal{R}_B .

Since $i_B(U)i_B(V) = i_B(Y)i_B(X)$ we'll have that

$$(L'_0(u; \alpha) - \zeta B)(L'_0(v; \beta) - \zeta B) = (L'_0(y; \beta) - \zeta B)(L'_0(x; \alpha) - \zeta B)$$

or equivalently $L_0(u_1, u_2, u_3; \alpha)L_0(v_1, v_2, v_3; \beta) = L_0(y_1, y_2, y_3; \beta)L_0(x_1, x_2, x_3; \alpha)$ which means that $L_0(x_1, x_2, x_3; \alpha)$ is a Lax matrix for this YB map. (Alternatively we can prove the YB property from proposition 2.1 by showing that the equation $L_0(x'; \alpha)L_0(y'; \beta)L_0(z'; \gamma) = L_0(x; \alpha)L_0(y; \beta)L_0(z; \gamma)$ implies $x' = x$, $y' = y$ and $z' = z$).

The proof for the other maps is similar. \square

In correspondence with remark 3.2, when we refer to YB maps on $\mathbb{C}^3 \times \mathbb{C}^3$ we mean on the open and dense domain of it that are defined (respectively for $\mathbb{C}^2 \times \mathbb{C}^2$).

Now since equation (6) has unique solution when $f_0(U) = f_0(X) = \alpha_1$, $f_1(U) = f_1(X) = \alpha_2$, $f_0(V) = f_0(Y) = \beta_1$ and $f_1(V) = f_1(Y) = \beta_2$, according to the above, the next corollary holds.

Corollary 4.4. *The equation*

$$L(u_1, u_2; \alpha_1, \alpha_2)L(v_1, v_2; \beta_1, \beta_2) = L(y_1, y_2; \beta_1, \beta_2)L(x_1, x_2; \alpha_1, \alpha_2) \quad (18)$$

is uniquely solvable with respect to u_1, u_2, v_1, v_2 . The mapping $R_{\alpha, \beta}(x, y) = (u, v)$ with vector parameters $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2)$, variables $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $u = (u_1, u_2)$, $v = (v_1, v_2)$ the unique solution of (18), is the symplectic parametric Yang-Baxter map (17) of Prop. 4.3.

Remark 4.5. By setting $\alpha_1 = \beta_1 = k$, the symplectic YB map of 4.3 yields the symplectic parametric YB map $R_{\alpha_2 \beta_2}$ with Lax matrix $L(x_1, x_2; \alpha_2) := L(x_1, x_2; k, \alpha_2)$. Here k is not a dynamical parameter as α_2 but just a free parameter. We have analogous result by setting $\alpha_2 = \beta_2$. If we set $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$ then we derive the trivial solution $U = Y$, $V = X$. That holds because this is the only solution of Eq.(6) with $f_0(U) = f_0(Y)$ and $f_1(U) = f_1(Y)$.

4.3 Classification by normal forms

Equation (6) is invariant under conjugation i.e:

$$P^{-1}(U - \zeta B)PP^{-1}(V - \zeta B)P = P^{-1}(Y - \zeta B)PP^{-1}(X - \zeta B)P$$

The same holds also for the Casimir functions f_0, f_1 , since $f_0(A)$, $f_1(A)$ are the coefficients of $\det(A - \zeta B)$. This means that we can restrict our attention to the next Jordan normal forms for the matrix B :

$$i) \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad ii) \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \quad iii) \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

More precisely let B_0 be one of these normal forms and

$$\tilde{R}_{\alpha,\beta}(x, y) = Q \circ \mathcal{R}_{B_0}(L'_0(x_1, x_2; \alpha_1, \alpha_2), L'_0(y_1, y_2; \beta_1, \beta_2)),$$

the symplectic YB map of Prop 4.3 with corresponding Lax matrix $L_0(a_1, a_2; c_1, c_2)$. Let also B a similar matrix with B_0 , $B = PB_0P^{-1}$. Then the map

$$R_{\alpha,\beta}(x, y) = Q(P[\mathcal{R}_{B_0}(P^{-1}L'_0(x_1, x_2; \alpha_1, \alpha_2)P, P^{-1}L'_0(y_1, y_2; \beta_1, \beta_2)P)]P^{-1}), \quad (19)$$

is a symplectic YB map with Lax matrix $L(a_1, a_2; c_1, c_2) = PL_0(a_1, a_2; c_1, c_2)P^{-1}$ and is exactly the unique solution of (18) with respect to $u = (u_1, u_2)$, $v = (v_1, v_2)$. (The first and the last multiplication by P and P^{-1} respectively of (19) is done at each factor of the cartesian product $\mathcal{S}_B(\alpha_1, \alpha_2) \times \mathcal{S}_B(\beta_1, \beta_2)$). Similar results hold for the Poisson maps $R_{\alpha,\beta}^0$ and $R_{\alpha,\beta}^1$ of Prop. 4.3.

If we are interested in real Lax matrices we have to include also the case where $B_0 = \begin{pmatrix} \lambda_1 & -\lambda_2 \\ \lambda_2 & \lambda_1 \end{pmatrix}$.

Example 4.6. We are going to apply the above results for $B = I$. In this case the non-zero Poisson brackets of \mathcal{L}_I are given by the relations:

$$\begin{aligned} \{a_{11}, a_{12}\} &= -a_{12}, \quad \{a_{11}, a_{21}\} = a_{21}, \quad \{a_{12}, a_{21}\} = a_{22} - a_{11}, \\ \{a_{12}, a_{22}\} &= -a_{12}, \quad \{a_{21}, a_{22}\} = a_{21}. \end{aligned}$$

The Casimir functions are $f_0(A) = \det A$, $f_1(A) = a_{11} + a_{22}$. If we set $f_0(A) = c_1$, $f_1(A) = c_2$, and solve with respect to a_{11} , a_{12} , for $a_{12} \neq 0$, we come up to the Lax matrix

$$L(a_1, a_2; c_1, c_2) = \begin{pmatrix} a_1 - \zeta & a_2 \\ \frac{a_1(c_2 - a_1) - c_1}{a_2} & c_2 - a_1 - \zeta \end{pmatrix},$$

where a_1 , a_2 here denotes a_{11} , a_{12} respectively. According to 4.4 the unique solution of the equation $L(u_1, u_2; \alpha_1, \alpha_2)L(v_1, v_2; \beta_1, \beta_2) = L(y_1, y_2; \beta_1, \beta_2)L(x_1, x_2; \alpha_1, \alpha_2)$ for any $\zeta \in \mathbb{C}$, gives the parametric YB map on $\mathbb{C}^2 \times \mathbb{C}^2$:

$$R_{\alpha,\beta}((x_1, x_2), (y_1, y_2)) = ((u_1, u_2), (v_1, v_2)) = Q \circ \mathcal{R}_I(L'(x_1, x_2; \alpha_1, \alpha_2), L'(y_1, y_2; \beta_1, \beta_2))$$

where $L'(x_1, x_2; \alpha_1, \alpha_2) = L(x_1, x_2; \alpha_1, \alpha_2) + \zeta I$. This map is symplectic with respect to the reduced symplectic structure defined by :

$$\{x_1, x_2\} = -x_2, \quad \{y_1, y_2\} = -y_2, \quad \{x_i, y_j\} = 0, \quad i = 1, 2,$$

on the corresponding symplectic leave $= \{(L(x_1, x_2; \alpha_1, \alpha_2), L(y_1, y_2; \beta_1, \beta_2)) \mid x_1, y_1 \in \mathbb{C}, x_2, y_2 \in \mathbb{C}^*\}$ of $\mathcal{L}_I \times \mathcal{L}_I$.

We can restrict matrix A to $SL_2(\mathbb{C})$ by setting $f_0(A) = 1$, $f_1(A) = c$. In this case the corresponding Lax matrix will be $M(a_1, a_2; c) = L(a_1, a_2; 1, c)$. Now the unique solution of the equation $M(u_1, u_2; \alpha)M(v_1, v_2; \beta) = M(y_1, y_2; \beta)M(x_1, x_2; \alpha)$ gives the parametric YB map:

$$R'_{\alpha,\beta}((x_1, x_2), (y_1, y_2)) = Q \circ \mathcal{R}_I(L'(x_1, x_2; 1, \alpha), L'(y_1, y_2; 1, \beta)).$$

5 Non-quadrirational Yang-Baxter maps

Non-quadrirational YB maps arise when the constant matrix B of \mathcal{L}_B is non-invertible. In this section we show how, in some cases, we can obtain non-quadrirational parametric YB maps as limits of the quadrirational maps of the previous section.

We consider invertible constant matrices $B = B(\varepsilon)$ depending on a parameter ε such that $\lim_{\varepsilon \rightarrow 0} \det B = 0$, and construct the quadrirational symplectic YB map $R(\varepsilon)$ with the corresponding Lax matrix $L(\varepsilon)$ of Prop. 4.3. By taking the limit of $R(\varepsilon)$ for $\varepsilon \rightarrow 0$ we derive a rational non-quadrirational YB map on $\mathbb{C}^2 \times \mathbb{C}^2$. This map is symplectic with the symplectic form that is induced by taking the limit of the structure matrix $J_{B(\varepsilon)}(L(\varepsilon) + \zeta B(\varepsilon))$. We restrict our analysis to the Jordan normal forms.

5.1 The Adler-Yamilov map

Consider $B = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}$. The Casimir functions on \mathcal{L}_B will be

$$f_0(A) = \det A, \quad f_1(A) = a_{11}\varepsilon + a_{22}$$

We set $f_0(A) = c$, $f_1(A) = 1$ and solve with respect to a_{11} , a_{22} :

$$a_{22} = 1 - \varepsilon a_{11}, \quad a_{11} = \frac{1 - \sqrt{1 - 4\varepsilon(c + a_{12}a_{21})}}{2\varepsilon}$$

Now we can construct the Lax matrix

$$L(a_1, a_2; c, 1) = \begin{pmatrix} \frac{1 - \sqrt{1 - 4\varepsilon(c + a_1a_2)}}{2\varepsilon} - \zeta & a_1 \\ a_2 & \frac{1}{2}(1 + \sqrt{1 - 4\varepsilon(c + a_1a_2)}) - \varepsilon\zeta \end{pmatrix} \quad (20)$$

where $a_1 = a_{12}$, $a_2 = a_{21}$. According to the previous section, the unique solution of the equation

$$L(u_1, u_2; \alpha, 1)L(v_1, v_2; \beta, 1) = L(y_1, y_2; \beta, 1)L(x_1, x_2; \alpha, 1)$$

will be $u_1 = u_{12}$, $u_2 = u_{21}$, $v_1 = v_{12}$, $v_2 = v_{21}$ where u_{ij} , v_{ij} are the corresponding elements of the matrices:

$$U = (f_2(X)YX - f_0(X)B^2)(f_2(X)(YB + BX) - f_1(X)B^2)^{-1}B \\ V = B^{-1}(YB + BX - UB)$$

by setting $X = L'(x_1, x_2; \alpha, 1)$ and $Y = L'(y_1, y_2; \beta, 1)$ (from definition $L' = L + \zeta B$). Here $f_2(X) = \det B = \varepsilon$, $f_1(X) = 1$ and $f_0(X) = \alpha$. This solution gives the quadrirational parametric YB map:

$$R_{\alpha\beta}((x_1, x_2), (y_1, y_2)) = ((u_1, u_2), (v_1, v_2)).$$

Now if we take the limit of u_i, v_i , $i = 1, 2$, for $\varepsilon \rightarrow 0$, we derive

$$\bar{u}_1 = \lim_{\varepsilon \rightarrow 0} u_1 = y_1 - \frac{(a-b)x_1}{1+x_1y_2}, \quad \bar{u}_2 = \lim_{\varepsilon \rightarrow 0} u_2 = y_2, \\ \bar{v}_1 = \lim_{\varepsilon \rightarrow 0} v_1 = x_1, \quad \bar{v}_2 = \lim_{\varepsilon \rightarrow 0} v_2 = x_2 + \frac{(a-b)y_2}{1+x_1y_2}$$

and the parametric YB map $\bar{R}_{\alpha\beta}((x_1, x_2), (y_1, y_2)) = ((\bar{u}_1, \bar{u}_2), (\bar{v}_1, \bar{v}_2))$. The latter is a map related to the Nonlinear *Schrödinger* systems [3, 11].

The induced symplectic structure is derived from (16) by taking the limit for $\varepsilon \rightarrow 0$ of $J_B(L'(x_1, x_2; \alpha, 1))$ and $J_B(L'(y_1, y_2; \beta, 1)) : \{x_1, x_2\} = 1$, $\{y_1, y_2\} = 1$, $\{x_i, y_j\} = 0$ i.e. the canonical symplectic form. The YB map $\bar{R}_{\alpha\beta}$ is symplectic with respect to this form.

5.2 A lift of the KdV quadgraph equation

Now we consider $B = \begin{pmatrix} \varepsilon & 1 \\ 0 & \varepsilon \end{pmatrix}$. In this case the Casimir functions on \mathcal{L}_B will be

$$f_0(A) = \det A, \quad f_1(A) = \varepsilon(a_{11} + a_{22}) - a_{21}.$$

We set again $f_0(A) = c$, $f_1(A) = 1$ and solve with respect to a_{21} , a_{12} :

$$a_{21} = \varepsilon(a_{11} + a_{22}) - 1, \quad a_{12} = \frac{a_{11}a_{22} - c}{\varepsilon(a_{11} + a_{22}) - 1}.$$

The Lax matrix will be

$$L(a_1, a_2; c, 1) = \begin{pmatrix} a_1 + \varepsilon\zeta & \frac{a_1 a_2 - c}{\varepsilon(a_1 + a_2) - 1} + \zeta \\ \varepsilon(a_1 + a_2) - 1 & a_2 + \varepsilon\zeta \end{pmatrix}, \quad (21)$$

where here a_1, a_2 denote a_{11}, a_{22} respectively. As before the unique solution of the equation

$$L(u_1, u_2; \alpha, 1)L(v_1, v_2; \beta, 1) = L(y_1, y_2; \beta, 1)L(x_1, x_2; \alpha, 1)$$

will be the elements u_{11}, u_{22} and v_{11}, v_{22} of the matrices (7) (denoted by u_1, u_2 and v_1, v_2 respectively), where $X = L'(x_1, x_2; \alpha, 1)$ and $Y = L'(y_1, y_2; \beta, 1)$. Here $f_2(X) = \det B = \varepsilon^2$. So we derive the corresponding quadrirational YB map $R_{\alpha\beta}((x_1, x_2), (y_1, y_2)) = ((u_1, u_2), (v_1, v_2))$, with Poisson brackets

$$\{x_1, x_4\} = -1 + \varepsilon(x_1 + x_4), \{y_1, y_4\} = -1 + \varepsilon(y_1 + y_4), \{x_i, y_j\} = 0. \quad (22)$$

By taking the limits of u_1, u_2 and v_1, v_2 , for $\varepsilon \rightarrow 0$, we derive the parametric Yang-Baxter map $\bar{R}_{\alpha\beta}((x_1, x_2), (y_1, y_2)) = ((\bar{u}_1, \bar{u}_2), (\bar{v}_1, \bar{v}_2))$ where

$$\bar{u}_1 = y_1 + \frac{\alpha - \beta}{x_1 + y_2}, \bar{u}_2 = y_2, \bar{v}_1 = x_1, \bar{v}_2 = x_2 - \frac{\alpha - \beta}{x_1 + y_2} \quad (23)$$

This map is symplectic with respect to the induced symplectic form defined by the limit of (22):

$$\{x_1, x_2\} = -1, \{y_1, y_2\} = -1, \{x_i, y_j\} = 0.$$

We are going to show that this can be squeezed down to the KdV quadgraph equation. We perform, first, the following change of variables: $x_2 \mapsto -x_2, y_2 \mapsto -y_2, \bar{u}_2 \mapsto -\bar{u}_2, \bar{v}_2 \mapsto -\bar{v}_2$ so

$$\bar{u}_1 = y_1 + \frac{\alpha - \beta}{x_1 - y_2}, \bar{u}_2 = y_2, \bar{v}_1 = x_1, \bar{v}_2 = x_2 + \frac{\alpha - \beta}{x_1 - y_2}$$

Notice now that if $y_1 = x_2$ then $\bar{u}_1 = \bar{v}_2$ and labeling the variables as $y_1 = x_2 = f, \bar{u}_1 = \bar{v}_2 = f_{12}, \bar{v}_1 = x_1 = f_1, y_2 = \bar{u}_2 = f_2$ both first and last equations reduce to the KdV quadgraph equation

$$(f_{12} - f)(f_1 - f_2) = \alpha - \beta.$$

This is the reason why (23) can be thought as a lift of KdV quadgraph equation. Actually this is an instance of the fact that all quadgraph equations of the ABS classification in [1] can be lifted to a 2-field quadgraph equation that can be cast into YB map form [12].

Remark 5.1. The limits of the Lax matrices (20) and (21) for $\varepsilon \rightarrow 0$ are

$$L_1(a_1, a_2; c) = \begin{pmatrix} a_1 a_2 + c - \zeta & a_1 \\ a_2 & 1 \end{pmatrix}, \quad L_2(a_1, a_2; c) = \begin{pmatrix} a_1 & c - a_1 a_2 - \zeta \\ -1 & a_2 \end{pmatrix}$$

respectively. These matrices are Lax matrices of the non-quadrirational YB maps of 5.1 and 5.2 respectively but the equation $L_2(u_1, u_2; \alpha)L_2(v_1, v_2; \beta) = L_2(y_1, y_2; \beta)L_2(x_1, x_2; \alpha)$ is not uniquely solvable with respect to u_i, v_i . Therefore we cannot derive the YB map (23) directly from the Lax matrix $L_2(a_1, a_2; c)$.

6 Conclusion

We saw how through matrix re-factorization and linear algebra considerations (namely Caley-Hamilton theorem) one is guided to consider the Casimirs of the Sklyanin bracket as the main conditions for a non-trivial solution of equation (6). We conjecture that a formula analogous to (7) can be found in the case of $n \times n$ matrices ($n > 2$).

A Proof of the YB property of the map given by (7)

We give a detailed proof of the Yang–Baxter property of the map of 3.1. Let X, Y, Z be generic elements of $GL_2(\mathbb{C})$. Because of Prop. 2.1, it suffices to show that the equation

$$(X' - \zeta B)(Y' - \zeta B)(Z' - \zeta B) = (X - \zeta B)(Y - \zeta B)(Z - \zeta B) \quad (24)$$

with $\det(X' - \zeta B) = \det(X - \zeta B)$ and $\det(Y' - \zeta B) = \det(Y - \zeta B)$ implies that $X' = X$, $Y' = Y$ and $Z' = Z$.

If we set $XYZ = K$, $XYB + XBZ + BYZ = L$ and $XB^2 + BYB + B^2Z = M$ then from equation (24) we derive the system :

$$X'Y'Z' = K, \quad X'Y'B + X'BZ' + BY'Z' = L, \quad X'B^2 + BY'B + B^2Z' = M$$

Last system implies

$$(X'B^{-1})^3 B^3 - (X'B^{-1})^2 M + X'B^{-1} L = K. \quad (25)$$

Since $\det(X' - \zeta B) = \det(X - \zeta B)$ we have that

$$f_2(X)(X'B^{-1})^2 - f_1(X)(X'B^{-1}) + f_0(X)I = 0 \quad (26)$$

and by evaluating the powers of $X'B^{-1}$, equation (25) gives:

$$X'B^{-1}[f_2^2 L - f_2 f_1 M + (f_1^2 - f_2 f_0) B^3] = f_2^2 K - f_2 f_0 M + f_1 f_0 B^3$$

(to alleviate notation we have dropped the X dependence from $f_i(X)$ and denote it simply by f_i , for $i = 0, 1, 2$). The last equation can be written as

$$X'B^{-1}[f_2^2 L - f_2 f_1 M + (f_1^2 - f_2 f_0) B^3] = XB^{-1}[f_2^2 L - f_2 f_1 M + (f_1^2 - f_2 f_0) B^3] + A \quad (27)$$

where

$$A = f_2^2 K - f_2 f_0 M + f_1 f_0 B^3 - XB^{-1}[f_2^2 L - f_2 f_1 M + (f_1^2 - f_2 f_0) B^3].$$

Replacing again K, L, M by $XYZ, XYB + XBZ + BYZ, XB^2 + BYB + B^2Z$ respectively we can factorize A as follows

$$A = (f_2(XB^{-1})^2 - f_1 XB^{-1} + f_0)(f_1 B^3 - f_2 B^2 Z - f_2 BYB)$$

and from Cayley–Hamilton theorem we get that $A = 0$. So from (27), for the generic elements $X, Y, Z \in GL_2(\mathbb{C})$ such that $\det[f_2^2 L - f_2 f_1 M + (f_1^2 - f_2 f_0) B^3] \neq 0$ we have that $X' = X$. In a similar way we can prove that $Z' = Z$ and finally $Y' = Y$ follows as well.

References

- [1] Adler V.E., Bobenko A.I., Suris Yu.B. [2003] Classification of integrable equations on quad-graphs. The consistency approach. Comm. Math. Phys. 233, 513-543.
- [2] Adler V.E., Bobenko A.I., Suris Yu.B. [2004] Geometry of Yang-Baxter maps: pencils of conics and quadrirational mappings. Comm. Anal. Geom. 12, 967-1007.
- [3] Adler V.E., Yamilov R.I. [1994] Explicit auto-transformations of integrable chains. J. Phys. A: Math. Gen. 27, 477-492
- [4] Bobenko A.I., Suris Yu.B. [2002] Integrable systems on quad-graphs. Int. Math. Res. Notices, 2002, No. 11, 573611.

- [5] Drinfeld V.G. [1992] On some Unsolved Problems in Quantum Group Theory. Lecture Notes in Math. 1510, 1-8.
- [6] Etingof P., Schedler T. and Soloviev A. [1999] Set-theoretical solutions to the quantum Yang-Baxter equation *Duke Math. J.* **100** no. 2 169–209
- [7] Etingof P. [2003] Geometric crystals and set-theoretical solutions to the quantum Yang-Baxter equation *Comm. Algebra* **31** no. 4 1961–1973
- [8] Lu J.-H., Yan M., Zhu Y.-C. [2000] On the set-theoretical Yang-Baxter equation. *Duke Math. J.* 104, 1-18.
- [9] Nijhoff F.W. [2002] Lax pair for the Adler (lattice Krichever-Novikov) system, *Phys. Lett. A* **297** 49–58
- [10] Papageorgiou V.G., Tongas A.G., Veselov A.P. [2006] Yang-Baxter maps and symmetries of integrable equations on quad-graphs. *J. Math. Phys.* 47, 083502 1-16.
- [11] Papageorgiou V.G., Tongas A.G., [2007] Yang-Baxter maps and multi-field integrable lattice equations. *J. Phys. A* 40, no. 42, 12677–12690.
- [12] Papageorgiou V.G., Tongas A.G., [2008] Yang-Baxter maps associated to elliptic curves (in preparation).
- [13] Reshetikhin N., Veselov A.P. [2005] Poisson Lie groups and Hamiltonian theory of the Yang-Baxter maps. *math.QA/0512328*.
- [14] Sklyanin E.K. [1983] Some algebraic structures connected with the Yang-Baxter equation. *Funct. Anal. Appl.* 16, No 4, 263-270.
- [15] Sklyanin E.K. [1988] Classical limits of $SU(2)$ -invariant solutions of the Yang-Baxter equation. *J. Soviet Math.* 40, No 1, 93-107.
- [16] Suris Yu.B., Veselov A.P. [2003] Lax pairs for Yang-Baxter maps. *J. Nonlin. Math. Phys.* 10, suppl.2, 223-230.
- [17] Veselov A.P. [2003] Yang-Baxter maps and integrable dynamics. *Phys. Lett. A*, 314, 214-221.
- [18] Veselov A.P. [2007] Yang-Baxter maps: dynamical point of view, *Combinatorial Aspects of Integrable Systems* (Kyoto, 2004), *MSJ Mem.* vol 17, pp 145-67.
- [19] Weinstein A., Xu P. [1992] Classical solutions to the Quantum Yang-Baxter equation, *Commun. Math. Phys.* 148, 309-343.